

Bifurcations in Fluctuating Systems: The Center-Manifold Approach

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Bifurcations in fluctuating dynamical systems are studied using the ideas of center-manifold reduction. The method provides not only a systematic procedure for the reduction of the system to a small number of variables—but also a classification scheme for the different kinds of dynamical behavior possible near bifurcation points. The joint probability density factorizes into a stationary Gaussian density $p(v/u)$ in the fast variables v , and a time-dependent density $P(u, t)$ in the slow variables u describing the dynamics on the center manifold $v = v_0(u)$. $P(u, t)$ obeys a reduced Fokker–Planck equation that can be written in a normal form by means of local nonlinear transformations. Both additive and multiplicative white noise are considered, as is colored noise. The results extend and formalize Haken's concept of adiabatic elimination of fast variables.

KEY WORDS: Fluctuations; Fokker–Planck equations; bifurcations; normal forms.

1. INTRODUCTION

An understanding of the behavior of systems with many degrees of freedom usually requires a reduction in the mathematical complexity of the full problem. Often certain variables can be eliminated as being unimportant or irrelevant, with the essential physics confined to the behavior of the remaining variables. This is a basic tenet of statistical mechanics. Although considerable work has been devoted to effecting explicit reductions both in deterministic and stochastic systems,^(1–8) much work remains to be done before the connection with nonequilibrium statistical mechanics is understood in detail. The methods all share a common theme: some unimportant variables adjust on a very short time scale, but the qualitatively important

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dynamics occurs on a much longer time scale. Haken⁽¹⁾ has stated, pictur-
esquely, that the fast-relaxing degrees of freedom are “enslaved” by the
more slowly evolving ones. In this case, one expects that the fast variables
may be eliminated by a method that has been termed “adiabatic elimina-
tion”.⁽¹⁾

In the deterministic case, these physically motivated approximations
find rigorous justification in bifurcation theory.^(9,10) When the solution to a
system of differential equations changes in a fundamental way at some
critical parameter value, the system is said to undergo a bifurcation. The
center-manifold theorem⁽¹¹⁾ characterizes the local behavior of solutions
near a bifurcation, and provides a justification for the reduction of the
dynamics to a relatively simple ordinary differential equation called a
“normal form.” The structure of the normal form depends only on the
number and nature of the eigenvalues of the linearized problem involved in
the bifurcation, the symmetries of the system, and certain nondegeneracy
conditions. The normal form approach thus provides us not only with a
lower-order equation, but also a *classification scheme* for local behavior of
systems near their bifurcation points.

In the present paper we extend the ideas of the center-manifold
reduction to systems subject to external noise. The effects of perturbations
are of greatest importance near a bifurcation point of a dynamical system.
It is precisely for this case that our method is derived. The point here is that
the insights provided by bifurcation theory for deterministic systems form a
secure basis for our intuition when noise is present. However, we do not
claim to make a mathematically rigorous extension of center-manifold
theory. Rather, our viewpoint is that bifurcation theory provides a natural
framework for tackling certain problems involving noisy perturbations.

Starting from the Fokker–Planck equation corresponding to a system
of Langevin equations, we find that—near a bifurcation—a reduced Fok-
ker–Planck equation is sufficient to describe the dynamics after a short
relaxation time. This reduced equation is, of course, easier to study than the
original problem, and corresponds to the normal form of the deterministic
system. Whole classes of starting equations lead to the same reduced
Fokker–Planck equation, so that again a classification scheme suggests
itself. Furthermore, bifurcations involving only a few eigenvalues are ex-
pected to be typical on physical grounds. This justifies, in retrospect, the
attention given to phenomenological one- and two-dimensional mod-
els^(12–25): they describe the dynamics near such bifurcations in higher-
dimensional systems.

Section 2 introduces the relevant aspects of bifurcation theory. This is
followed in Section 3 by a description of the reduction scheme for the case
of additive white noise. The results of this section essentially verify the

results of earlier investigations,⁽¹⁾ although we are able to provide a more complete description of the conditional probability distribution of the fast variables, and its dependence on the slow variables on the center manifold. New results are presented in Section 4, where the reduction procedure is applied to the case of multiplicative noise. In Section 5, a specific example is discussed which displays a noise-dependent bifurcation.⁽¹⁷⁾ The results are compared with those obtained earlier in another parameter regime,⁽¹⁸⁾ and a physical interpretation of the differences is offered. Finally, in Section 6 we briefly discuss a way of using our method to model the effects of nonwhite noise. Our main results are summarized in Section 7.

2. BIFURCATIONS IN DETERMINISTIC SYSTEMS

In this section we briefly describe the method of reduction of a deterministic system to a lower-dimensional form in the neighborhood of a bifurcation point. We emphasize those aspects of the theory that can be taken over to the stochastic problem.

An autonomous n -dimensional ($n \leq \infty$) dynamical system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mu) \quad (2.1)$$

where μ is a parameter, has an equilibrium point $\mathbf{x}_0(\mu)$ if $\mathbf{F}(\mathbf{x}_0, \mu) = 0$. Without loss of generality, we may take $\mathbf{x}_0 = 0$. The linear stability of this point is determined by the eigenvalues of the linearization matrix $\mathbf{M}(\mu) \equiv \nabla \mathbf{F}|_{\mathbf{x}=0}$. The equilibrium point undergoes a *bifurcation* as μ is varied, when one or several eigenvalues cross (transversally) the imaginary axis in the complex plane. The linearized problem can be put into Jordan normal form by going into the eigenbasis; at the bifurcation point μ_0 , we may write (2.1) in the form

$$\dot{\mathbf{u}} = \mathbf{A}(\mu_0)\mathbf{u} + \mathbf{f}(\mathbf{u}, \mathbf{v}, \mu_0) \quad (2.2a)$$

$$\dot{\mathbf{v}} = \mathbf{B}(\mu_0)\mathbf{v} + \mathbf{g}(\mathbf{u}, \mathbf{v}, \mu_0), \quad (2.2b)$$

where the matrices \mathbf{A} and \mathbf{B} have eigenvalues λ such that $\text{Re} \lambda_A(\mu_0) = 0$, $\text{Re} \lambda_B(\mu_0) \neq 0$, and \mathbf{f}, \mathbf{g} are at least quadratic in \mathbf{u}, \mathbf{v} . For simplicity, we shall assume that all eigenvalues of \mathbf{B} have negative real part. This assumption is not necessary, but allows us to consider the physically interesting case of a stable equilibrium losing stability.

2.1. The Center Manifold

At the bifurcation, one or several eigenvalues are on the imaginary axis. Their eigenvectors span a plane that is tangent at $\mathbf{x} = 0$ to an invariant surface called the *center manifold* (Fig. 1). The remaining eigenvalues have

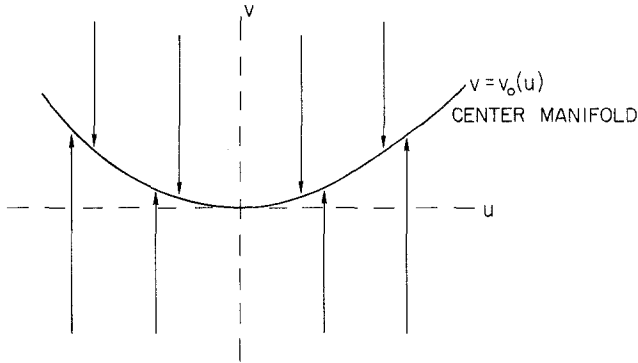


Fig. 1. The center manifold. The fast variables contract onto the center manifold on a fast time scale. The dynamics on the center manifold is governed by nonlinear terms only, and takes place on a slow time scale.

negative real parts, and sufficiently close to $\mathbf{x} = 0$ they dominate the dynamics in directions perpendicular to the center manifold. As a result, trajectories starting off the center manifold contract onto it on a *rapid* time scale, and the subsequent evolution of the system (2.1) occurs on the center manifold. Here the approach to an asymptotic state is governed by the nonlinear terms, and hence near the origin it occurs on a *long* time scale. For example, at a Hopf bifurcation [Eq. (2.11)], the *amplitude* r of the neutrally stable oscillation evolves as a result of the cubic nonlinearity.

The center manifold can be obtained as a power series in \mathbf{u} ,

$$\mathbf{v} = \mathbf{v}_0(\mathbf{u}) \quad (2.3)$$

by an iterative procedure.⁽²⁶⁾ Consequently, after a short transient, the dynamics of the system is described, with exponentially small errors, by the system

$$\dot{\mathbf{u}} = \mathbf{A}(\mu_0)\mathbf{u} + f(\mathbf{u}, \mathbf{v}_0(\mathbf{u}), \mu_0) \quad (2.4)$$

with $f(\mathbf{u}, \mathbf{v}_0(\mathbf{u}), \mu_0)$ given as a power series in \mathbf{u} . Since typically only one, or at most several eigenvalues cross the unit circle simultaneously, Eq. (2.4) represents a drastic reduction in the dimension of the original system (2.1). This reduction is justified rigorously by the center-manifold theorem.⁽¹¹⁾ The geometry of the reduction is illustrated in Fig. 1.

As an example, consider a second-order system, with an equilibrium point at the origin, and suppose that we have a bifurcation at which \mathbf{M} has one zero eigenvalue and one negative eigenvalue. In suitable coordinates we

then have

$$\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} au^2 + buv + cv^2 \\ du^2 + euv + fv^2 \end{pmatrix} + O(3) \quad (2.5)$$

where λ is of order unity, and $O(3)$ indicates cubic or higher-order terms in u, v . The center manifold is defined by the condition $\dot{\Delta} = 0$, where $\Delta \equiv v - v_0(u)$. It follows that

$$v_0(u) = \frac{d}{\lambda} u^2 + O(u^3) \quad (2.6)$$

and the dynamics on the center manifold is given by

$$\dot{u} = au^2 + O(u^3) \quad (2.7)$$

Note that in the case $a \neq 0$ the dynamics do not depend on the curvature of the center manifold.

2.2. Normal Forms

For multiple bifurcations (i.e., those involving several eigenvalues) the analysis can be taken a stage further, and Eq. (2.4) simplified by means of a near identity transformation

$$\mathbf{w} = \mathbf{u} + \mathbf{h}(\mathbf{u}) \quad (2.8)$$

where $\mathbf{h} = O(\mathbf{u}^2)$. This is because the reduction is local. The power series $\mathbf{h}(\mathbf{u})$ is chosen such that order by order as many of the nonlinear terms in (2.4) are removed as possible. The terms that can be removed depend on the structure of the matrix $\mathbf{A}(\mu_0)$ and any symmetries respected by the nonlinear terms. The resulting equation

$$\dot{\mathbf{w}} = \mathbf{A}(\mu_0)\mathbf{w} + \mathbf{k}(\mathbf{w}, \mu_0) \quad (2.9)$$

is called a *normal form* of the system (2.1) at the bifurcation point μ_0 . All problems with the same matrix \mathbf{A} and the same symmetry properties, satisfying the same nondegeneracy conditions, can be reduced to the *same* normal form by appropriate transformations (2.8).

We consider two examples. In the case where there is a pair of pure imaginary eigenvalues (Hopf bifurcation), \mathbf{A} has the Jordan form

$$A = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \quad (2.10)$$

and by means of appropriate transformations (2.8) the dynamics at the bifurcation can be put in the form

$$\begin{aligned} \dot{r} &= ar^3 + O(r^5) \\ \dot{\theta} &= \omega + br^2 + O(r^4) \end{aligned} \quad (2.11)$$

The next simplest case involves two zero eigenvalues. The center manifold is again two dimensional. In the case

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

the dynamics on the center manifold is given by equations of the form

$$\begin{aligned} \dot{u} &= v + au^2 + buv + cv^2 + O(3) \\ \dot{v} &= du^2 + evv + fv^2 + O(3) \end{aligned} \quad (2.12)$$

To put these equations into normal form we introduce the near identity transformation

$$\begin{aligned} u &= \bar{u} + \alpha u^2 + \beta uv + \gamma v^2 + O(3) \\ v &= \bar{v} + \delta u^2 + \epsilon uv + \eta v^2 + O(3) \end{aligned} \quad (2.13)$$

Choosing $\delta = -a$, $2\alpha - \epsilon = b$, $\beta - \eta = c$, $\epsilon = f$, we obtain the normal form

$$\begin{aligned} \dot{\bar{u}} &= \bar{v} + O(3) \\ \dot{\bar{v}} &= A\bar{u}^2 + B\bar{u}\bar{v} + O(3) \end{aligned} \quad (2.14)$$

with $A = d$ and $B = e + 2a$.

2.3. Unfolding

The final stage in the analysis involves a consideration of the dynamics for parameter values *near* the bifurcation point $\mu = \mu_0$. An application of the center-manifold theorem to system (2.2) augmented with the equation $\dot{\mu} = 0$, shows that if the equilibrium is preserved, then the dynamics is given by

$$\dot{\mathbf{w}} = \mathbf{A}(\mu)\mathbf{w} + \mathbf{k}(\mathbf{w}, \mu) \quad (2.15)$$

with negligible error. In other words, the nonlinear terms can be evaluated at the bifurcation point μ_0 (provided that none of the leading nonlinear terms vanish there). This completes the reduction of the system (2.1) near a bifurcation to a standard low-dimensional normal form.

The following four examples are of interest:

(i) *The Transcritical Bifurcation:*

$$\dot{u} = (\mu - \mu_c)u + au^2 + O(u^3) \quad (2.16)$$

(ii) *The Pitchfork Bifurcation.* This bifurcation typically obtains when the basic system is symmetrical under reflection:

$$\dot{u} = (\mu - \mu_c)u + au^3 + O(u^5) \quad (2.17)$$

If $a < 0$, the bifurcation is supercritical (i.e., it gives rise to two new stable nontrivial states).

(iii) *The Hopf Bifurcation.* Here the amplitude r of the oscillations near the bifurcation obeys [cf. example (ii) above]

$$\dot{r} = (\mu - \mu_c)r + ar^3 + O(r^5) \tag{2.18}$$

(iv) *A Codimension-Two Bifurcation.* Here a second parameter ν has been adjusted to give two zero eigenvalues when $\mu = \mu_0$. To capture the generic dynamics near such a bifurcation it is necessary to change ν from its critical value so that as μ is increased two eigenvalues pass through zero in close succession. The normal form becomes

$$\begin{aligned} \dot{\bar{u}} &= \bar{v} + O(3) \\ \dot{\bar{v}} &= (\mu - \mu_0)\bar{u} + (\nu - \nu_0)\bar{v} + A\bar{u}^2 + B\bar{u}\bar{v} + O(3) \end{aligned} \tag{2.19}$$

Since the unfolding contains two parameters, we shall call this bifurcation a codimension-two bifurcation.

The dynamics described by the normal forms for different problems can now be analyzed once and for all. The normal forms thus classify the possible dynamics near bifurcations and the conditions under which they occur. In a specific problem only the coefficients in $\mathbf{k}(\mathbf{w})$ have to be computed; the form of \mathbf{k} is fixed by the matrix \mathbf{A} and the symmetries of the basic problem (2.1). In the following sections we shall show how the above ideas generalize to noisy systems.

3. REDUCTION IN THE PRESENCE OF ADDITIVE NOISE

We now present a reduction procedure for systems subject to additive noise. Since the application of our method is entirely analogous for the other cases mentioned in Section 2, we give the details for only the transcritical bifurcation. The pitchfork, Hopf, and codimension-two examples are discussed only briefly.

3.1. Transcritical Bifurcation

It is sufficient to apply our reduction procedure to a second-order system—generalization to higher-order systems is straightforward.

As before, assume the origin is a fixed point of the deterministic system. Then

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \tilde{L} \begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{N}(x, y) + \begin{pmatrix} 0 \\ \xi \end{pmatrix} \tag{3.1}$$

where \tilde{L} is a constant 2×2 matrix, \mathbf{N} is a vector of nonlinear functions, and ξ is Gaussian white noise of intensity 2κ ,

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = 2\kappa\delta(t-t') \quad (3.2)$$

At the bifurcation point, a linear change of coordinates puts the system in the form [see Eq. (2.5)]

$$\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} au^2 + buv + cv^2 + O(3) \\ du^2 + euv + fv^2 + O(3) \end{pmatrix} + \begin{pmatrix} \sigma\xi \\ \tau\xi \end{pmatrix} \quad (3.3)$$

where σ, τ, λ are constants of order unity ($\lambda > 0$).

The object of interest is the joint probability density, which obeys the Fokker-Planck equation corresponding to (3.3):

$$\begin{aligned} \partial_t \mathcal{P}(u, v, t) = & -\partial_u \{ [au^2 + buv + cv^2 + O(3)] \mathcal{P} \} \\ & -\partial_v \{ [-\lambda v + du^2 + euv + fv^2 + O(3)] \mathcal{P} \} \\ & + \kappa\sigma^2 \partial_{uu}^2 \mathcal{P} + 2\kappa\sigma\tau \partial_{uv}^2 \mathcal{P} + \kappa\tau^2 \partial_{vv}^2 \mathcal{P} \end{aligned} \quad (3.4)$$

We want to isolate the behavior of \mathcal{P} near the deterministic center manifold, which in the present case is given by equation (2.6). To do this, we first introduce the conditional density $p(v, t|u)$ and the (marginal) density $P(u, t)$:

$$\mathcal{P}(u, v, t) = p(v, t|u) \cdot P(u, t) \quad (3.5)$$

After an initial relaxation time of order λ^{-1} , we expect the joint density to be confined to a narrow strip, peaked about the center manifold. We thus assume that p has the time-independent form

$$p(v, t|u) = \left[\frac{\Lambda(u)}{\pi} \right]^{1/2} \exp\{ -\Lambda(u) \cdot [v - v_0(u)]^2 \} \quad (3.6)$$

which is a normalized Gaussian in the variable v , with a width that depends on u . The self-consistency of this assumption is demonstrated in the Appendix.

By substituting (3.5) and (3.6) into (3.4) and integrating over v on a narrow strip centered on the center manifold, we obtain an equation for

$P(u, t)$:

$$\begin{aligned} \partial_t P(u, t) = & -\partial_u \left[(au^2 + buv_0 + cv_0^2 + O(3))P \right] \\ & - \left[au^2 + buv_0 + cv_0^2 + O(3) \right] \frac{\Lambda'}{2\Lambda} P \\ & - \left[-\lambda + eu + 2fv_0 + O(2) \right] P - 2\kappa\tau^2 \Lambda P + 4\kappa\sigma\tau \Lambda v_0' P \\ & + \kappa\sigma^2 \left[\partial_{uu}^2 P + \frac{\Lambda'}{\Lambda} \partial_u P + \left(\frac{\Lambda''}{2\Lambda} - \frac{\Lambda'^2}{4\Lambda^2} - 2\Lambda v_0'^2 \right) P \right] \end{aligned} \quad (3.7)$$

where the primes denote differentiation with respect to u . In deriving (3.7), we treated the sharply peaked Gaussian (3.6) as a δ function. We will return to this point later.

We emphasize that our analysis is a local one. Thus, integration of the full Fokker–Planck equation generates several “surface terms”. Previous work along these lines⁽¹⁾ assumed that the Langevin equations were globally valid, and consequently ignored these terms.

By adjusting the function Λ , we are able to reduce (3.7) so that it becomes an equation of continuity for $P(u, t)$. Physically, we then have a conserved flow of probability within a band about the center manifold, provided we allow the width of the band to vary.

To achieve this, expand Λ in powers of u :

$$\Lambda(u) = \Lambda_0 + \Lambda_1 u + O(u^2) \quad (3.8)$$

where Λ_0 and Λ_1 are constants. We next introduce scaled variables in order to explicitly display the relative size of terms appearing in (3.7). With ϵ a small quantity, we let

$$\begin{aligned} u &= \epsilon \hat{u}, & \kappa &= \epsilon^3 \hat{\kappa} \\ \epsilon^3 \Lambda_0 &= \hat{\Lambda}_0, & \epsilon^3 \Lambda_1 &= \hat{\Lambda}_1 \end{aligned} \quad (3.9)$$

Next, we substitute (3.8) and (3.9) into (3.7), and ignore terms of order ϵ^2 . Thus,

$$\begin{aligned} \partial_t P(\epsilon \hat{u}, t) = & \epsilon \left\{ -\partial_{\hat{u}} \left[a\hat{u}^2 P \right] + \hat{\kappa}\sigma^2 \partial_{\hat{u}\hat{u}}^2 P \right\} \\ & + \left\{ \lambda - 2\hat{\kappa}\tau^2 \hat{\Lambda}_0 + \epsilon \left[\frac{8\hat{\kappa}\sigma\tau \hat{\Lambda}_0 d}{\lambda} - e - 2\hat{\kappa}\tau^2 \hat{\Lambda}_1 \right] \hat{u} \right\} P \end{aligned} \quad (3.10)$$

To ensure that P satisfies an equation of continuity, we demand that terms

proportional to P cancel. This requires that

$$0 = \lambda - 2\kappa\tau^2\Lambda_0 \quad (3.11)$$

and

$$\Lambda_1 = \frac{2\sigma d}{\kappa\tau^3} - \frac{\epsilon}{2\kappa\tau^2} \quad (3.12)$$

Equation (3.11) expresses the balance between the fast drift toward the center manifold and a “diffusion pressure” provided by the noise.

With these choices for Λ_0 and Λ_1 , we are left with a reduced Fokker–Planck equation:

$$\partial_t P(u, t) = -\partial_u[au^2P] + \kappa\sigma^2\partial_{uu}^2P \quad (3.13)$$

Just as in the deterministic analysis, we can rederive the reduced dynamics when the system is not precisely at the bifurcation, but nearly so. Introducing the small unfolding parameter $\mu \sim O(\epsilon)$, the reduced Fokker–Planck equation now includes a linear drift,

$$\partial_t P(u, t) = -\partial_u[(\mu u + au^2)P] + \kappa\sigma^2\partial_{uu}^2P \quad (3.14)$$

which is the noisy analog of equation (2.16). Equations (3.11) and (3.12) remain unchanged—together with (3.14), they are the major results of this section. An alternative derivation is given in the Appendix.

Before turning to other cases, we would like to return to the integration of the full Fokker–Planck equation (3.4). From (3.6), we see that the conditional density is

$$p(v|u) = \left(\frac{\Lambda}{\pi}\right)^{1/2} \exp\{-\Lambda \cdot [v - v_0]^2\} \quad (3.15)$$

According to the theory of distributions,⁽²⁷⁾ in the limit $\kappa \downarrow 0$ (3.15) acts (rigorously) like a δ function. This justifies the integration leading to Eq. (3.7).

3.2. Pitchfork Bifurcation

If we assume that our system is symmetric under reflection, then instead of (3.3) we have

$$\frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -\lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} au^3 + bu^2v + cuv^2 + dv^3 + O(5) \\ eu^3 + fu^2v + guv^2 + hv^3 + O(5) \end{bmatrix} + \begin{bmatrix} \sigma\xi \\ \tau\xi \end{bmatrix} \quad (3.16)$$

In this case the center manifold for the deterministic system is given by

$$v = v_0(u) = \frac{e}{\lambda} u^3 + O(5) \quad (3.17)$$

The derivation of the appropriate reduced Fokker–Planck equation is only slightly different than for the transcritical bifurcation. The Gaussian inverse width is now

$$\Lambda(u) = \Lambda_0 + \Lambda_2 u^2 + O(u^4) \tag{3.18}$$

and the proper scaling is

$$\begin{aligned} u &= \epsilon \hat{u}, & \kappa &= \epsilon^4 \hat{\kappa} \\ \epsilon^4 \Lambda_0 &= \hat{\Lambda}_0, & \epsilon^4 \Lambda_2 &= \hat{\Lambda}_2 \end{aligned} \tag{3.19}$$

Proceeding as in the previous section, we now find that for the pitchfork

$$\Lambda_0 = \lambda / (2\kappa\tau^2) \tag{3.20}$$

$$\Lambda_2 = \frac{3\sigma e}{\kappa\tau^3} - \frac{f}{\kappa\tau^2} \tag{3.21}$$

and the reduced Fokker–Planck equation is

$$\partial_t P(u, t) = -\partial_u [(\mu u + au^3)P] + \kappa\sigma^2 \partial_{uu}^2 P \tag{3.22}$$

where the unfolding parameter μ is now an order ϵ^2 quantity. Equation (3.22) is the noisy analog of (2.17).

3.3. Hopf Bifurcation

For a Hopf bifurcation, the center manifold is two dimensional. Consequently, to apply the reduction procedure, we must start from a system which is at least third order. The reduced Fokker–Planck equation will involve two independent variables, plus time.

For example, a third-order system at a Hopf bifurcation can be put in the form

$$\frac{d}{dt} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} F \\ G \\ H \end{pmatrix} + \xi \begin{pmatrix} \sigma \\ \tau \\ \zeta \end{pmatrix} \tag{3.23}$$

For simplicity, we have assumed that the original system has only a single additive-noise forcing term. The Fokker–Planck equation corresponding to (3.23) is reduced by assuming that the joint probability density is

$$\mathcal{P}(u, v, w, t) = \left[\frac{\Lambda(u, v)}{\pi} \right]^{1/2} \exp\{-\Lambda \cdot [w - w_0]^2\} P(u, v, t) \tag{3.24}$$

where $w = w_0(u, v)$ is the equation of the center manifold. The inverse-width function Λ may be chosen so that $P(u, v, t)$ obeys the reduced

Fokker–Planck equation

$$\begin{aligned} \partial_t P(u, v, t) &= -\partial_u [(-\omega v + a_1 u^2 + b_1 uv + c_1 v^2 + d_1 u^3 + e_1 u^2 v + f_1 uv^2 + g_1 v^3)P] \\ &\quad -\partial_v [(\omega u + a_2 u^2 + b_2 uv + c_2 v^2 + d_2 u^3 + e_2 u^2 v + f_2 uv^2 + g_2 v^3)P] \\ &\quad + \kappa [\sigma^2 \partial_{uu}^2 P + 2\sigma\tau \partial_{uv}^2 P + \tau^2 \partial_{vv}^2 P] \end{aligned} \quad (3.25)$$

where a_1, \dots, g_2 are constants appearing in the power series expansions of F and G .

It remains to be seen whether the Fokker–Planck equation (3.25) can be simplified by means of transformations of the form (2.8). This is discussed in more detail in the following section, where it is found that the near identity transformation (2.8) transforms the Fokker–Planck equation to the Fokker–Planck equation for the normal form. For the Hopf bifurcation, it is possible to remove the quadratic terms in u and v entirely—it is for this reason that (3.25) includes the cubic terms in the drift. Consequently, (3.25) becomes, in appropriate variables r, θ ,

$$\begin{aligned} \partial_t \bar{P}(r, \theta, t) &= -\frac{1}{r} \partial_r [(\mu r^2 + ar^4)\bar{P}] - \partial_\theta [(\omega + br^2)\bar{P}] + \kappa(\sigma \cos \theta + \tau \sin \theta)^2 \partial_{rr}^2 \bar{P} \\ &\quad + \kappa(-\sigma^2 \cos \theta \sin \theta + 2\sigma\tau \cos^2 \theta + \tau^2 \sin \theta \cos \theta) \partial_r \left[\frac{1}{r} \partial_\theta \bar{P} \right] \\ &\quad + \kappa \left[-\sigma^2 \frac{\sin \theta}{r} \partial_\theta \cos \theta - \frac{2\sigma\tau \sin \theta}{r} \partial_\theta \sin \theta + \frac{\tau^2 \cos \theta}{r} \partial_\theta \sin \theta \right] \partial_r \bar{P} \\ &\quad + \kappa \left[-\frac{\sigma^2}{r^2} \sin \theta \partial_\theta \sin \theta - \frac{2\sigma\tau \sin \theta}{r^2} \partial_\theta \cos \theta + \frac{\tau^2 \cos \theta}{r^2} \partial_\theta \cos \theta \right] \partial_\theta \bar{P} \end{aligned} \quad (3.26)$$

This equation may be further reduced to a one-dimensional Fokker–Planck equation by eliminating the angular dependence. The rapid rotation in the θ direction suggests that the physically interesting evolution will be described by a purely radial density $P(r, t)$. This may be accomplished by introducing the conditional probability $p(\theta/r)$ by

$$\bar{P}(r, \theta, t) = p(\theta/r) \cdot P(r, t) \quad (3.27)$$

as in Eq. (3.5). It can be shown that (cf. Ref. 5) the appropriate conditional probability is uniform. The final equation for $P(r, t)$ is

$$\partial_t P(r, t) = -\frac{1}{r} \partial_r [(\mu r^2 + ar^4)P] + \kappa \frac{\sigma^2 + \tau^2}{2} \partial_{rr}^2 P \quad (3.28)$$

which should be compared with (2.18).

3.4. A Codimension Two Bifurcation

In this section we describe in a little more detail the reduction procedure for a codimension-two bifurcation problem in the presence of white noise. The elimination of the fast variables proceeds exactly along the lines already described. However, since at a codimension-two bifurcation the center manifold is two dimensional, the reduced Fokker–Planck equation describing the dynamics on the center manifold will involve two variables, plus time. In the deterministic problem we saw how, with the aid of near identity nonlinear transformations, the dynamics on the center manifold could be put into as simple a form as possible—the normal form (cf. Section 2). We would like to generalize this idea to stochastic bifurcation problems. This implies that we have to perform such transformations on the Fokker–Planck equation.

We start with the three-dimensional system, in Jordan form:

$$\frac{d}{dt} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} F \\ G \\ H \end{pmatrix} + \xi(t) \begin{pmatrix} \sigma \\ \tau \\ \zeta \end{pmatrix} \tag{3.29}$$

On the center manifold the dynamics is described by the reduced Fokker–Planck equation for the system (2.12):

$$\begin{aligned} \partial_t P(u, v, t) = & -\partial_u [(v + au^2 + buv + cv^2)P] - \partial_v [(du^2 + euv + fv^2)P] \\ & + \kappa\sigma^2\partial_{uu}^2 P + 2\kappa\sigma\tau\partial_{uv}^2 P + \kappa\tau^2\partial_{vv}^2 P \end{aligned} \tag{3.30}$$

with $\kappa = O(3)$. The transformation (2.13), applied to this equation, generates a Fokker–Planck equation for the density $\bar{P}(\bar{u}, \bar{v})$ defined by

$$P(u, v) = \bar{P}(\bar{u}, \bar{v}) \cdot \partial(\bar{u}, \bar{v})/\partial(u, v) \tag{3.31}$$

To the same accuracy as (3.30), \bar{P} obeys the equation

$$\begin{aligned} \partial_t \bar{P}(\bar{u}, \bar{v}, t) = & -\partial_{\bar{u}}(\bar{v}\bar{P}) - \partial_{\bar{v}}[(d\bar{u}^2 + (e + 2a)\bar{u}\bar{v})\bar{P}] \\ & + \kappa\sigma^2\partial_{\bar{u}\bar{u}}^2 \bar{P} + 2\kappa\sigma\tau\partial_{\bar{u}\bar{v}}^2 \bar{P} + \kappa\tau^2\partial_{\bar{v}\bar{v}}^2 \bar{P} \end{aligned} \tag{3.32}$$

i.e., the normal form of the Fokker–Planck equation is the Fokker–Planck equation for the normal form. We conclude that a classification of the stochastic bifurcation problem follows entirely the classification of the deterministic problems. Analogous results hold for the unfolded normal forms.

4. REDUCTION IN THE PRESENCE OF MULTIPLICATIVE NOISE

The above reduction procedure may be extended to the case of multiplicative noise. In this section, this is done explicitly for the pitchfork

bifurcation. The results are applied to a concrete example in Section 5. The calculations for other kinds of bifurcation are entirely analogous, and we omit the details.

We begin with a second-order system at a bifurcation point which has been put into the form

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & -\lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} au^3 + bu^2v + cuv^2 + dv^3 + O(5) \\ eu^3 + fu^2v + guv^2 + hv^3 + O(5) \end{bmatrix} \\ &+ \xi(t) \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \end{aligned} \quad (4.1)$$

where the k_{ij} are constants. This may be compared with (3.16) for the additive noise problem. We have assumed that the white noise ξ multiplies the linear terms of the dynamic system. Observe that as the origin is approached, the effect of noise diminishes. We expect therefore that the joint probability density will be correspondingly more sharply peaked about the center manifold. This is indeed what we find.

The Fokker–Planck equation is

$$\begin{aligned} \partial_t \mathcal{P}(u, v, t) &= -\partial_u [(au^3 + bu^2v + cuv^2 + dv^3 + O(5))\mathcal{P}] \\ &- \partial_v [(-\lambda v + eu^3 + fu^2v + guv^2 + hv^3 + O(5))\mathcal{P}] \\ &+ \kappa \{ \partial_u [(k_{11}u + k_{12}v) \{ \partial_u [(k_{11}u + k_{12}v)\mathcal{P}] + \partial_v [(k_{21}u + k_{22}v)\mathcal{P}] \}] \\ &\quad + \partial_v [(k_{21}u + k_{22}v) \{ \partial_u [(k_{11}u + k_{12}v)\mathcal{P}] \\ &\quad \quad + \partial_v [(k_{21}u + k_{22}v)\mathcal{P}] \}] \} \end{aligned} \quad (4.2)$$

where 2κ is the intensity of noise [see Eq. (3.2)]. As before, we factor $\mathcal{P}(u, v, t)$ as

$$\mathcal{P}(u, v, t) = \left[\frac{\Lambda(u)}{\pi} \right]^{1/2} \exp\{-\Lambda \cdot [v - v_0]^2\} \cdot P(u, t) \quad (4.3)$$

where $v = v_0(u)$ is the center manifold for the deterministic problem, given by [see Eq. (3.17)]

$$v_0(u) = \frac{e}{\lambda} u^3 + O(u^5) \quad (4.4)$$

To obtain an equation for $P(u, t)$, we integrate over v :

$$\begin{aligned}
 \partial_t P(u, t) = & -\partial_u \left[(au^3 + bu^2v_0 + cuv_0^2 + dv_0^3 + O(5))P \right] \\
 & + \kappa \partial_u \left\{ \left[(k_{11} + 2k_{22} - 2k_{12}v'_0)(k_{11}u + k_{12}v_0) + k_{12}(k_{21}u + k_{22}v_0) \right. \right. \\
 & \quad \left. \left. + \frac{\Lambda'}{\Lambda} (k_{11}u + k_{12}v_0)^2 \right] P + (k_{11}u + k_{12}v_0)^2 P' \right\} \\
 & + \left[\lambda - (fu^2 + 2guv_0 + 3hv_0^2 + O(4)) \right. \\
 & \quad \left. - \frac{\Lambda'}{2\Lambda} (au^3 + bu^2v_0 + cuv_0^2 + dv_0^3 + O(5)) \right] P \\
 & + \kappa \left\{ (k_{11} + k_{22})^2 + [(3k_{11} + 2k_{22})(k_{11}u + k_{12}v_0) \right. \\
 & \quad \left. + k_{12}(k_{21}u + k_{22}v_0)] \frac{\Lambda'}{2\Lambda} \right. \\
 & \quad \left. + (k_{11}u + k_{12}v_0)^2 \left(-\frac{\Lambda''}{2\Lambda} + \frac{3}{4} \frac{\Lambda'^2}{\Lambda^2} - 2\Lambda v_0'^2 \right) \right. \\
 & \quad \left. + 4(k_{11}u + k_{12}v_0)(k_{21}u + k_{22}v_0)\Lambda v_0' \right. \\
 & \quad \left. - 2(k_{21}u + k_{22}v_0)^2\Lambda + 2(k_{11} + k_{12}v_0')^2 - k_{12}(k_{21} + k_{22}v_0') \right. \\
 & \quad \left. + 2(k_{11}u + k_{12}v_0) \left(k_{12}v_0'' - \frac{\Lambda'}{\Lambda} [k_{11} + k_{12}v_0'] \right) \right. \\
 & \quad \left. - (3k_{11} + 2k_{22})(k_{11} + k_{12}v_0') \right\} P \tag{4.5}
 \end{aligned}$$

where the primes denote differentiation with respect to u , and all quantities are evaluated at $v = v_0$. If we put

$$\Lambda(u) = \frac{1}{u^2} (\Lambda_0 + \Lambda_1 + \Lambda_2 u^2 + O(u^2)) \tag{4.6}$$

and introduce the scaled variables

$$\begin{aligned}
 u &= \epsilon \hat{u}, & \kappa &= \epsilon^2 \hat{\kappa} \\
 \epsilon^2 \Lambda_0 &= \hat{\Lambda}_0, & \epsilon^2 \Lambda_2 &= \hat{\Lambda}_2, & \Lambda_1 &= \hat{\Lambda}_1
 \end{aligned} \tag{4.7}$$

we find that we can eliminate the terms proportional to P appearing in (4.5)

to leading order by choosing

$$\Lambda_0 = \frac{\lambda}{2\kappa k_{21}^2} \tag{4.8}$$

$$\Lambda_1 = \frac{1}{2k_{21}^2} ([k_{11} - k_{22}]^2 - 2k_{12}k_{21}) \tag{4.9}$$

$$\Lambda_2 = \frac{1}{2\kappa k_{21}^2} \left(a - f + 2e \left[\frac{3k_{11} - k_{22}}{k_{21}} \right] \right) \tag{4.10}$$

As anticipated, the joint density (4.3) has an enhanced peak as compared with the additive noise counterpart [Eq. (3.6)]. The algebraic divergence in the probability density at the deterministic fixed point is a familiar feature of multiplicative noise problems,^(14,15,17,28) even when the noise is not δ -correlated.⁽¹⁹⁾ Note that when $u = 0$ is *fixed*, the conditional density again acts as a δ function as $\kappa \downarrow 0$.

To leading order, then, we are left with a reduced Fokker-Planck equation:

$$\begin{aligned} \partial_t P(u, t) = & -\partial_u [(\mu u + au^3)P] \\ & + \kappa \partial_u \{ k_{11}^2 u^2 P' + (-k_{11}^2 + 2k_{11}k_{22} + k_{12}k_{21})uP \} \end{aligned} \tag{4.11}$$

where the unfolding parameter $\mu \sim O(u^2)$ has been included.

5. AN EXAMPLE: THE VAN DER POL-DUFFING OSCILLATOR

In this section, we apply the above formalism to the van der Pol-Duffing oscillator

$$\ddot{x} = \alpha x + \beta \dot{x} + Ax^3 + Bx^2\dot{x} \tag{5.1}$$

As emphasized elsewhere,⁽¹⁸⁾ this equation is relevant to a wide variety of physical phenomena. We treat the problem for three cases: the deterministic system, the system with an additive noise forcing term, and the system with a (linear) multiplicative noise term.

After writing down the proper equations for the “reduced dynamics,” we will focus on the long-time behavior. In the deterministic case, the local analysis using the center-manifold theorem properly describes the asymptotic behavior provided the initial conditions are sufficiently close to the fixed point. However, when noise fluctuations are present, the situation becomes somewhat more subtle. In general, the global characteristics may be important for a complete description of a dynamical system.

For example, consider a deterministic system with two stable stationary points. Each point will have its own *basin of attraction* such that trajectories inside a basin will asymptotically approach that fixed point.

These basins do not overlap—once inside a particular fixed point’s basin of attraction, the system cannot leave. Consequently, a local analysis can determine *all* the relevant dynamics for a limited region of phase space. However, if noise is added, there can be transitions *between* basins. These transitions will occur on some characteristic time scale T —for low noise intensities, T will be large.

As we have emphasized, our local analysis is expected to be valid after some short relaxation time of order λ^{-1} . In general, however, we cannot expect the reduced Fokker–Planck equation to be valid for times comparable to T . Consequently, we have focused on deriving an *evolution* equation for the probability density. However, there are cases when the asymptotic solution of the reduced Fokker–Planck equation is of physical interest. For example, this is the case when the global problem contains only one stationary point. Alternatively, T may be so large that the dynamics on the center manifold effectively reaches its asymptotic form in some characteristic time t_∞ , satisfying

$$\lambda^{-1} \ll t_\infty \ll T$$

5.1. Deterministic Analysis

We rewrite Eq. (5.1) as a system of first-order equations, with $y = \dot{x}$:

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ Ax^3 + Bx^2y \end{pmatrix} \tag{5.2}$$

We take A , B , and β to be negative constants of order unity, and consider the bifurcation at $\alpha = 0$. Then (5.2) describes the equation of motion for a unit mass subject to nonlinear damping moving in a quartic potential. As α changes sign, the potential changes from a single-well to a double-well potential, which is obviously a physically interesting regime.

It is easy to transform (5.2) so that the linearized problem is in diagonal form at the bifurcation point:

$$\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} au^3 + bu^2v + cuv^2 + dv^3 \\ -au^3 - bu^2v - cuv^2 - dv^3 \end{pmatrix} \tag{5.3}$$

where

$$\begin{aligned} a &= -A/\beta \\ b &= -[B + 3A/\beta] \\ c &= -[2B + 3A/\beta] \\ d &= -[B + A/\beta] \quad \text{and} \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x - y/\beta \\ y/\beta \end{pmatrix} \end{aligned} \tag{5.4}$$

The equation for the center manifold is (cf. Section 2)

$$v = \frac{a}{\beta} u^3 + O(u^5) \quad (5.5)$$

and the dynamics on the center manifold is

$$\dot{u} = au^3 + O(u^5) \quad (5.6)$$

When $\alpha \neq 0$, (5.2) has a *small* nonzero eigenvalue

$$\mu = \frac{\beta}{2} - \left(\frac{\beta^2}{4} + \alpha \right)^{1/2} \simeq -\alpha/\beta \quad (5.7)$$

and using this as an unfolding parameter, we arrive at the reduced dynamical equation

$$\dot{u} = -\frac{\alpha}{\beta} u - \frac{A}{\beta} u^3 + O(u^5) \quad (5.8)$$

The asymptotic state of the system will be given by the stable stationary points of (5.8):

$$u(t \rightarrow \infty) = \begin{cases} 0, & \text{when } \alpha \leq 0 \\ \pm \left(\frac{\alpha}{A} \right)^{1/2}, & \text{when } \alpha > 0 \end{cases} \quad (5.9)$$

The situation is depicted in Fig. 2—the name “pitchfork bifurcation” is seen to be appropriate.

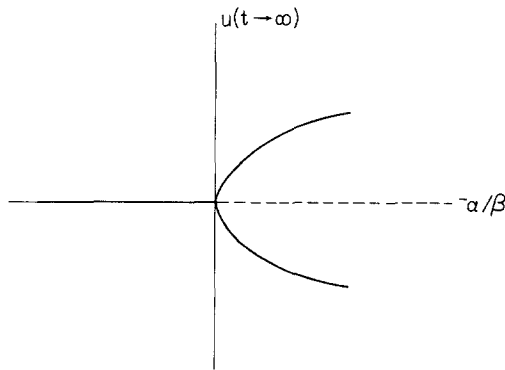


Fig. 2. The pitchfork bifurcation. The stationary points of (5.8) are plotted as a function of α . —, stable; ---, unstable.

5.2. Additive Noise

We now examine the system

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ Ax^3 + Bx^2y \end{pmatrix} + \begin{pmatrix} 0 \\ \xi \end{pmatrix} \quad (5.10)$$

At the bifurcation point $\alpha = 0$, a linear change of variables yields

$$\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} au^3 + bu^2v + cv^2 + dv^3 \\ -au^3 - bu^2v - cv^2 - dv^3 \end{pmatrix} + \xi \begin{pmatrix} -1/\beta \\ +1/\beta \end{pmatrix} \quad (5.11)$$

where (5.4) still holds. The reduced Fokker–Planck equation follows immediately from Section 3.2:

$$\partial_t P(u, t) = +\partial_u \left[\left(\frac{\alpha}{\beta} u + \frac{A}{\beta} u^3 \right) P \right] + \frac{\kappa}{\beta^2} \partial_{uu}^2 P \quad (5.12)$$

[see Eq. (3.22)]. This equation admits the stationary solution

$$P_0(u, t) = N \cdot \exp \left\{ \frac{-\beta}{\kappa} \left(\alpha u^2 + \frac{A}{2} u^4 \right) \right\} \quad (5.13)$$

where N is a normalization constant. This function changes from single to double peaked as α increases past zero (Fig. 3). The peaks are given by Eq. (5.9). Since the joint probability density $\mathcal{P}(u, v, t)$ is just (5.13) multiplied by a function peaked about the center manifold, the maxima of the full density function coincide with the stable stationary points of the deterministic system.

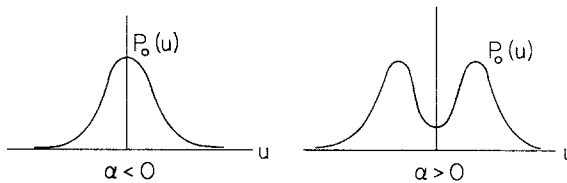


Fig. 3. The stationary probability density (5.13) in the case of additive noise. (a) $\alpha < 0$, (b) $\alpha > 0$.

5.3. Multiplicative Noise

We now take α in Eq. (5.1) to fluctuate about its mean value:

$$\alpha \rightarrow \alpha + \xi(t)$$

and consider the system of equations

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ Ax^3 + Bx^2y \end{pmatrix} + \begin{pmatrix} 0 \\ \xi x \end{pmatrix} \quad (5.14)$$

The same linear change of variables as before yields

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} au^3 + bu^2v + cuv^2 + dv^3 \\ -au^3 - bu^2v - cuv^2 - dv^3 \end{pmatrix} \\ &+ \frac{\xi}{\beta} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \end{aligned} \quad (5.15)$$

at the bifurcation point. The reduced Fokker–Planck equation is [see (4.11)]

$$\partial_t P(u, t) = +\partial_u \left[\left(\frac{\alpha}{\beta} u + \frac{A}{\beta} u^3 \right) P \right] + \frac{\kappa}{\beta^2} \partial_u \{ u^2 P' - 4uP \} \quad (5.16)$$

This admits the stationary density

$$P_0(u) = N \cdot |u|^{(4-\alpha\beta/\kappa)} \exp \left\{ -\frac{\beta A}{2\kappa} u^2 \right\} \quad (5.17)$$

The function defined by (5.17) has three qualitatively distinct behaviors depending on the value of α :

- (i) If $\alpha > 4\kappa/\beta$, $P_0(u=0) = 0$.
- (ii) If $4\kappa/\beta > \alpha > 5\kappa/\beta$, $P_0(u=0) \rightarrow \infty$, but (5.17) is normalizable, and thus a valid probability density.
- (iii) If $5\kappa/\beta > \alpha$, then P_0 is not normalizable. In this case, the divergence at $u = 0$ is too strong. Since system (5.15) admits the stationary solution

$$\mathcal{P}(u, v) = \delta(u)\delta(v) \quad (5.18)$$

we take this to be the appropriate probability density in this regime.

Figure 4 summarizes these results for multiplicative noise.

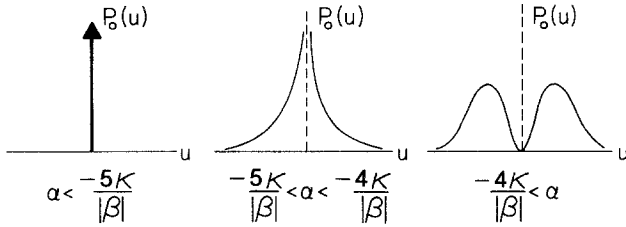


Fig. 4. The stationary probability density (5.17) in the case of multiplicative noise. The transition points between regimes depends explicitly on the noise intensity κ .

5.3. Discussion

Comparison of the deterministic and additive noise results for system (5.1) confirms our intuitive notions about small random perturbations. In both cases, the asymptotic solutions change qualitatively at the parameter value $\alpha = 0$. In line with the usual physical interpretation of statistical mechanics, the maxima of the probability density (5.13) correspond to the macroscopically observed states, and these coincide precisely with the stable fixed points of the deterministic system. This is an encouraging sign that we have developed a proper reduction scheme.

Some new features enter when we consider multiplicative noise perturbations. There are now three qualitatively distinct asymptotic solutions. Furthermore, the positions of the bifurcation points in parameter space separating these different solutions depend on the noise intensity. This is an example of a noise-dependent bifurcation in a two-dimensional system.

It had previously been established that the second-order system (5.14) displays a noise-dependent bifurcation in the weak damping limit.⁽¹⁸⁾ The present reduction procedure corresponds to a strong damping limit, and we find that the bifurcation points are now *shifted* toward more negative values of the “spring constant” $-\alpha$. The direction of this shift makes physical sense: the more heavily damped the system, the more difficult it is for the system to escape the fixed point at the origin. Escape from the origin requires either a large deterministic repelling force (more positive α), or a larger noise intensity.

One might imagine that we could avoid the careful reduction procedure we have presented by recognizing from the outset that it corresponds to the strongly damped limit of (5.14). After all, this is the standard justification for using the first-order Smoluchowski equation to describe Brownian motion of a particle in a potential.⁽²⁹⁾ A glance at Eq. (4.11), however, shows that all four entries in the “noise matrix” appearing in (4.1) contribute to the position of the noise-dependent bifurcation.

For example, had we analyzed the stochastic differential equation

$$0 = \beta \dot{x} + \alpha x + Ax^3 + Bx^2 \dot{x} + \xi x \quad (5.19)$$

[this is obtained from (5.14) by ignoring the inertial term], we would again find three distinct stationary solutions, but the bifurcation points would occur at $\alpha = 0$ and $\alpha = -\kappa/\beta$.

6. MODELING COLORED NOISE

Another useful application of the present reduction scheme is that it enables one to study the effects of noisy perturbations with finite correlation times. In contrast to δ -correlated white noise, this physically more realistic case is often referred to as "colored noise."

The importance of colored noise problems is obvious. In particular, one might wonder if the phenomenon of noise-dependent bifurcations is the result of the singular character of the white noise correlation function. (In the case of first-order stochastic differential equations, at least, it has been demonstrated that these transitions persist in the presence of finite noise correlation times.⁽¹⁹⁾)

The usual difficulty encountered is that a process such as

$$\dot{x} = \mu x - x^3 + \eta(t) \quad (6.1)$$

where $\eta(t)$ is not δ -correlated noise, does not describe a Markov process in x space. Since Markov processes are mathematically easier to handle, it has been suggested⁽¹⁴⁾ that $\eta(t)$ be regarded as the output of a process involving white noise input, for example,

$$\dot{\eta}(t) = -\lambda \eta + \xi(t) \quad (6.2)$$

where $\xi(t)$ is δ correlated. As is well known, (6.2) leads to η being the Ornstein-Uhlenbeck process.⁽¹⁴⁾ The point is that the system (6.1), (6.2) now describes a Markov process in (x, η) space.

The obvious complication is that one must now deal with a higher-dimensional phase space. However, under certain circumstances, the reduction procedure developed above may be applied to reduce the complexity of the Fokker-Planck equation.

To demonstrate this for the system (6.1), (6.2), we first write it in the standard form

$$\frac{d}{dt} \begin{pmatrix} x \\ \eta \end{pmatrix} = \begin{pmatrix} \mu & 1 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} -x^3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \xi \end{pmatrix} \quad (6.3)$$

A change of variables diagonalizes the linear problem:

$$\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \mu & 0 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} - \left[u + \frac{v}{\mu + \lambda} \right]^3 \\ 0 \end{pmatrix} + \xi \begin{pmatrix} (\mu + \lambda)^{-1} \\ -1 \end{pmatrix} \quad (6.4)$$

The system has a pitchfork bifurcation at $\mu = 0$, and the results of Section 3.2 are immediately relevant. In particular, we note that the joint density function is the product of a time-independent Gaussian centered on the center manifold and a slowly evolving function of u . The interesting information involves the dependence of the probability density on x , which can be obtained by making the change of variables back to (x, η) , and then integrating over η .

7. CONCLUSIONS

In this paper we have shown how the ideas of center-manifold reduction can be adapted to bifurcation problems in the presence of noise. The method, essentially suggested by Haken,⁽¹⁾ enables one to reduce a high-dimensional system to a low-dimensional one near a bifurcation point by eliminating the “fast variables.” We find that the joint probability density can be self-consistently factorized into a time-independent Gaussian conditional density for the fast variables and a time-dependent density for the slow variables. Self-consistency is assured by imposing the requirement that probability is conserved. This requires the inclusion of surface terms, representing the flux of probability out of the region in which a local analysis is valid, and demands that the width of the Gaussian density of the fast variables about the center manifold depend on the position on the center manifold, i.e., on the slow variables.

In each case we indicated the scaling necessary to provide an interesting competition between the deterministic dynamics and the noise (either additive or multiplicative), and the size of the corresponding domain in phase space within which our Fokker–Planck analysis is valid. In terms of the small bifurcation parameter μ , we find that for additive noise and the (a) transcritical bifurcation (Section 3.1), $\kappa = O(\mu^3)$, $u = O(\mu)$; (b) pitchfork bifurcation (Section 3.2), $\kappa = O(\mu^2)$, $u = O(\mu^{1/2})$; (c) Hopf bifurcation (Section 3.3), $\kappa = O(\mu^2)$, $r = O(\mu^{1/2})$, with analogous results for multiplicative noise.

The method we present is a systematic one. It enables us to obtain both the conditional density of the fast variables and the reduced Fokker–Planck equation for the slow variables. The concept of “normal forms” carries over to stochastic bifurcation problems, and enables us to classify

the possible dynamics near a bifurcation in terms of normal forms of the reduced Fokker–Planck equation. We have seen that to leading order in a local analysis these normal forms are simply the Fokker–Planck equations for the deterministic problems. The classification thus follows the deterministic classification except for the possibility of noise-dependent bifurcations. Many of the simple stochastic bifurcation problems studied thus far can be interpreted as describing the dynamics on the center manifold of an appropriate higher-dimensional system.

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APPENDIX

In this Appendix, we verify that the conditional density $p(v, t/u)$ derived in Section 3.1 is properly treated as independent of time. To do this, we assume that p has a time-dependent inverse-width $\Lambda(u, t)$, and then show that $\partial_t p$ is small enough that

$$\partial_t \mathcal{P}(u, v, t) = p(v, t | u) \cdot \partial_t P(u, t) \quad (\text{A1})$$

to leading order. It then follows that p may be approximated by the time independent form (3.6).

As in Section 3, we write the joint density \mathcal{P} as a product of the conditional density p and the marginal density P [see Eq. (3.5)]. The full Fokker–Planck equation (3.4) is

$$\begin{aligned} \partial_t [pP] = & -\partial_u [(au^2 + buv + cv^2 + O(3))pP] \\ & - P\partial_v [(-\lambda v + du^2 + ev + fv^2 + O(3))p] \\ & + \kappa\sigma^2\partial_{uu}^2 [pP] + 2\kappa\sigma\tau\partial_{uv}^2 [pP] + \kappa\tau^2 P\partial_{vv}^2 p \end{aligned} \quad (\text{A2})$$

We integrate this equation over all values of v . If p and $\partial_v p$ vanish sufficiently rapidly at infinity, we obtain an equation for $\partial_t P$:

$$\partial_t P(u, t) = -\partial_u [\langle f \rangle P] + \kappa\sigma^2\partial_{uu}^2 P \quad (\text{A3})$$

where

$$f \equiv au^2 + buv + cv^2 + \dots$$

and

$$\langle f \rangle \equiv \int_{-\infty}^{\infty} fp(v, t | u) dv \tag{A4}$$

Observe that all terms appearing in the expansion in (A4) must be included since the integration is not confined to a region near the origin.

We next assume that p is of the form

$$p(v, t | u) = \left[\frac{\Lambda(u, t)}{\pi} \right]^{1/2} \exp\{-\Lambda(u, t) \cdot [v - v_0]^2\} \tag{A5}$$

Substitution of (A3) and (A5) into (A2) yields

$$\begin{aligned} \dot{p}/p = & -\frac{1}{P} \partial_u [fP - \langle f \rangle P] - fQ(u, \eta) + \lambda - \partial_v g \\ & - 2\eta\Lambda(\lambda v - g) + \kappa\tau^2(4\Lambda^2\eta^2 - 2\Lambda) \\ & + 4\kappa\sigma\tau[-\Lambda\eta P'/P - \Lambda'\eta + \Lambda v'_0 - \Lambda\eta Q(u, \eta)] \\ & + \kappa\sigma^2 \left[\frac{2P'}{P} Q + Q^2 + \frac{\Lambda\Lambda'' - \Lambda'^2}{\Lambda^2} - \Lambda''\eta^2 + 4\Lambda'\eta v'_0 \right. \\ & \left. - 2\Lambda v_0'^2 + 2\Lambda\eta v_0'' \right] \end{aligned} \tag{A6}$$

where

$$\begin{aligned} Q(u, \eta) &= \frac{\Lambda'}{2\Lambda} - \Lambda'\eta^2 + 2\Lambda\eta v'_0 \\ \eta &= v - v_0(u) \\ g &= du^2 + euv + fv^2 + O(3) \end{aligned}$$

and primes and dots denote differentiation with respect to u and t , respectively. We regard u and η as independent expansion parameters, and thus write

$$\begin{aligned} f(u, \eta) &= au^2 + bu\eta + c\eta^2 + O(u^3, u^2\eta, \dots) \\ \lambda v - g &= \lambda\eta - eu\eta - h\eta^2 + O(u^3, u^2\eta, \dots) \end{aligned}$$

To order η^0 in Eq. (A6) we have

$$\begin{aligned} \frac{\dot{\Lambda}}{2\Lambda} = & -\frac{1}{P} \partial_u [au^2P - \langle f \rangle P] - \frac{au^2\Lambda'}{2\Lambda} + \lambda \\ & - eu + \kappa\sigma^2 \left[\frac{P'\Lambda'}{P\Lambda} + \frac{\Lambda'^2}{4\Lambda^2} + \frac{\Lambda\Lambda'' - \Lambda'^2}{\Lambda^2} - 2\Lambda v_0'^2 \right] \\ & + 4\kappa\sigma\tau\Lambda v'_0 - 2\kappa\tau^2\Lambda \end{aligned} \tag{A7}$$

Consider the scaling [cf. Eq. (3.9)]

$$\kappa \sim O(u^3) \sim \Lambda^{-1}, \quad \frac{\dot{\Lambda}}{\Lambda} \sim O(u^0)$$

We may then choose $\Lambda(u)$ so that $\dot{\Lambda}/\Lambda \sim O(u^2)$ in (A7)—in fact, *this requires that $\Lambda(u)$ is precisely that found in Section 3.1, Eqs. (3.8), (3.11), (3.12).* Consequently,

$$\dot{p}/p \sim O(\eta^1, u^2)$$

The reduced Fokker–Planck equation (A3) shows $\dot{P}/P \sim O(u)$. Restricting our analysis about the center manifold implies that $\eta \sim O(u^2)$, and we conclude that

$$\dot{P}/P \sim O(u), \quad \dot{p}/p \sim O(u^2)$$

thus validating assumption (A1).

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